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Approximate Calculation of Vortex Trajectories of Slender Bodies at Incidence

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Nomenclature

a	= body radius
K	= nondimensional constant defining growth of vortex strength with downstream distance
$\tilde{r}, r$	= dimensional and nondimensional distance between vortex line and body longitudinal axis, respectively
t	= time
U	= freestream velocity
U_δ	= velocity at boundary-layer edge
W	= undisturbed velocity in the crossflow plane ($= U\sin\alpha$)
x,y,z	= nondimensional coordinates (normalized by a)
$\tilde{x},\tilde{y},\tilde{z}$	= coordinates of vortex lines, defined in Fig. 1
α	= angle of attack
Γ	= vortex strength
Λ	= empirical factor relating vorticity shed from boundary layer to vorticity of the free vortices
ξ	= asymptotic angle of vortices
τ	= characteristic residence time

Introduction

THE aerodynamics of slender bodies at incidence is of fundamental importance to the analysis of high-performance missiles and aircraft. As a result, much work has gone into the understanding of this basically nonlinear situation. (Nielsen[1] has 100 references in a recent review.) A main cause of nonlinearity is the interaction between the shed vorticity and the body flowfield. In the present Note, a simple method for predicting the trajectory of vortices shed on the lee side of slender bodies, not requiring empirical information, is shown. When the angle of attack is less than 20 deg, the vorticity shed rolls up into one or more steady, symmetric pair of vortices on the lee side. In the crossflow plane, this can be pictured as flow around a cylinder with one or more pairs of symmetrically placed, counter-rotating vortices. This case will be examined in the present Note as it highlights all the main points of the model.

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Analysis

Figure 1 shows a typical slender body with one pair of shed vortices. For simplicity, we take the case of a circular cylinder with one pair only, but more vortices or other shapes can be accommodated by a simple generalization.[2,3] Incompressible, steady inviscid (except in the body boundary layer) flow is assumed. Under these circumstances, the flow in the crossflow plane (Fig. 1b) is described by Föppl's[4] classical solution, which can be used to relate the vortex strength and position (in a purely two-dimensional case). Applying the slenderness requirement, the flow in any section A can be approximated by the flow around an infinite cylinder with the local cross section. The relation between vortex strength and position is[5]

$$\Gamma = W\frac{(\tilde{r}^2 - a^2)^2(\tilde{r}+a^2)}{\tilde{r}^5} \tag{1}$$

where $W = U\sin\alpha$. Nondimensionalizing by dividing by the radius a, we obtain

$$\frac{\Gamma}{Ua} = \sin\alpha\frac{(r^2-1)^2(r^2+1)}{r^5} = \sin\alpha\left(r - \frac{1}{r} - \frac{1}{r^3} + \frac{1}{r^5}\right) \tag{2}$$

The measured vorticity in the wake is produced by feeding sheets resulting from the separation of boundary layers. The flux of vorticity added to the sheet per unit time and length is[1]

$$\frac{\partial\Gamma}{\partial t} = \Lambda\frac{U_\delta^2}{2} \tag{3}$$

where Λ is a constant (<1) which accounts for loss of vorticity due to mixing, dissipation, etc. For circular cylinders, its value is approximately 0.5.[6] The external speed at the edge of the boundary layer is $U_\delta \approx 2U\sin\alpha$, as separation is assumed to take place at roughly 90 deg from the front stagnation point. Thus, in a fixed-time period τ, the total vorticity shed in a given section is

$$\Gamma = \int_0^\tau 2\Lambda U^2 \sin^2\alpha \, dt \approx 2\Lambda\tau U^2\sin^2\alpha \tag{4}$$

We now apply the slender-body condition again with constant cross section (for simplicity). Each section is then

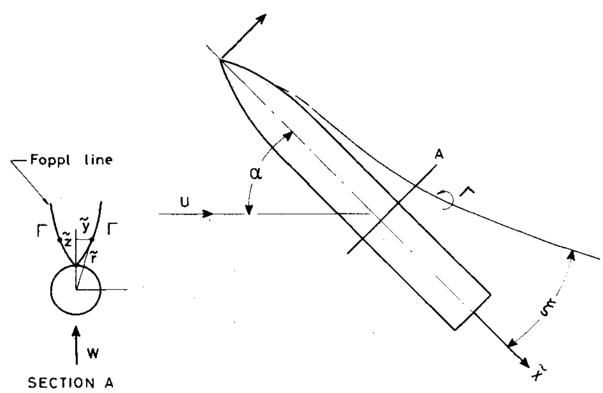

Fig. 1 Schematic description of flowfield and coordinate system.

producing approximately the same amount of vorticity (neglecting the interaction between the vortices in the wake and the boundary-layer development on the cylinder), so that

$$\Gamma \propto \tilde{x} \tag{5}$$

where $\tilde{x}$ is measured from the point of detachment of the vortex, or nondimensionally,

$$\Gamma/aU = Kx \tag{6}$$

where K is a yet undetermined constant. Its value will be obtained next.

For the observed steady-state vortex wake to be possible, the added vorticity shed [Eq. (3)] at each instant has to be exactly equal to the vorticity convected downstream in the $\tilde{x}$ direction.[6] This can be interpreted such that the vorticity at a plane in the wake, normal to the cylinder longitudinal axis, can be described by the vorticity produced over the upstream part of the cylinder that has reached that point by convection. The $\tilde{x}$ coordinate can be written as

$$\tilde{x} = \int_0^{\tau} U\cos\alpha \, dt = \tau U \cos\alpha \tag{7}$$

and substituting Eqs. (4) and (7) into Eq. (6), we obtain

$$K = 2\Lambda \frac{\sin^2\alpha}{\cos\alpha} \tag{8}$$

Now, substituting Eqs. (6) and (8) into Eq. (2), we obtain a relation between the nondimensional distance of the vortex from the cylinder section center r and the downstream distance x from the body leading edge.

$$2\Lambda x \tan\alpha = r - \frac{1}{r} - \frac{1}{r^3} + \frac{1}{r^5} \tag{9}$$

Discussion

Figure 2 shows the vortex trajectory as a function of distance along the slender body, as calculated from Eq. (9). The behavior at $r=1$ is probably not realistic as 1) viscous effects will dominate; and 2) to the present order of approximation, the body leading edge is not well defined.

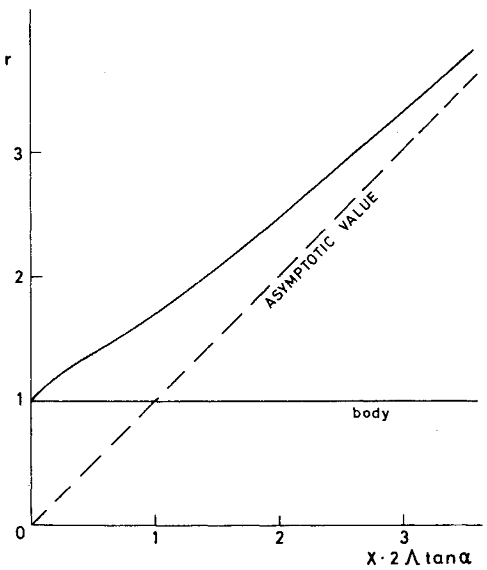

Fig. 2 Nondimensional distance of vortex from body centerline vs weighted nondimensional distance from point of vortex separation along centerline. Dashed line indicates asymptotic direction.

However, the rest of the trajectory is remarkably similar to experimentally obtained patterns.[1,7]

The vortex lines, as indicated in Fig. 2, tend to an asymptote as r (or x) goes to large values. This asymptotic angle can serve as a good test of the accuracy of the present method, as its values have been well documented. This angle, called ξ by Thomson and Morrison,[7] has been shown[7,8] to obey the relationship

$$\left[\frac{\tan\xi}{\tan\alpha}\right]_{\text{exp}} \simeq 0.8 \tag{10}$$

for low Mach numbers. Taking the limit of Eq. (9),

$$\lim_{x\to\infty} (r/x) = 2\Lambda\tan\alpha = \tan\xi \tag{11}$$

i.e., we immediately obtain the observed independence of the ratio [Eq. (10)] from the angle of attack. The experimental data leading to Eq. (10) were based on sideviews so that the distance r has to be separated into components. The Föppl line has an asymptote of 30 deg[5] for large distances so that $\lim_{r\to\infty} z = r\cos 30\text{deg} = 0.866r$. As a result,

$$\tan\xi = \lim_{x\to\infty} \frac{z}{x} = 1.732\Lambda\tan\alpha \tag{12}$$

Taking the value of $\Lambda = 0.5$, as previously mentioned, we see that

$$\left[\frac{\tan\xi}{\tan\alpha}\right]_{\text{theor}} = 0.866 \tag{13}$$

i.e., within less than 10% of the experimental value (actually, within the experimental spread of the data).

In view of the rough approximations involved in this analysis, the agreement with existing data is very encouraging, showing that the model reproduces observed data both qualitatively and quantitatively. The simplicity of the model, which enables calculation of the vortex shapes by hand, opens up possibilities for rapid engineering estimates of forces and moments on maneuvering slender bodies. The model can be generalized to include more than one pair of vortices [changing only Eq. (1)], to nonconstant cross sections ($K \neq$ const), asymmetric vortex positions, when the relation between the vortex strengths Γ and their equilibrium positions is known, etc. The latter problem, including also compressibility effects, which is relevant to bodies at $\alpha > 30$ deg, is being studied at the moments. This will be a first iteration in a more accurate calculation of forces and moments, and will be reported in due time.

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